

Hilberg Exponents: New Measures of Long Memory in the Process

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Abstract

The paper concerns the rates of power-law growth of mutual information computed for a stationary measure or for a universal code. The rates are called Hilberg exponents and four such quantities are defined for each measure and each code: two random exponents and two expected exponents. A particularly interesting case arises for conditional algorithmic mutual information. In this case, the random Hilberg exponents are almost surely constant on ergodic sources and are bounded by the expected Hilberg exponents. This property is a “second-order” analogue of the Shannon-McMillan-Breiman theorem, proved without invoking the ergodic theorem. It carries over to Hilberg exponents for the underlying probability measure via Shannon-Fano coding and Barron inequality. Moreover, the expected Hilberg exponents can be linked for different universal codes. Namely, if one code dominates another, the expected Hilberg exponents are greater for the former than for the latter. The paper is concluded by an evaluation of Hilberg exponents for certain sources such as the mixture Bernoulli process and the Santa Fe processes.

Keywords: mutual information, universal coding, Kolmogorov complexity, ergodic processes

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I Preliminaries and main results

According to a conjecture by Hilberg [1, 2], the mutual information between two adjacent long blocks of text in natural language grows like a power of the block length. This property strongly differentiates natural language from k -parameter sources, for which the mutual information is proportional to the logarithm of the block length [3, 4, 5]. In [6, 7] a class of stationary processes, called Santa Fe processes, has been exhibited, which feature the power-law growth of mutual information. Moreover, it was shown in [6] that Hilberg's conjecture implies Herdan's law, an integrated version of the famous Zipf's law in linguistics [8]. Later, Dębowski [9, 10] tested Hilberg's conjecture experimentally by approximating the mutual information with the Lempel-Ziv code [11] and a newly introduced universal code called switch distribution [10]. Whereas the estimates of mutual information for the Lempel-Ziv code grow roughly as a power law for both a k -parameter source and natural language, cf. [12], the other code does reveal the difference predicted by Hilberg's conjecture: the estimates of mutual information for the switch distribution grow as a power law for natural language whereas only logarithmically for a k -parameter source [13].

To provide more theory for Hilberg's conjecture, in this paper we abstract from its empirical verification and we investigate the bounding rates for power-law growth of mutual information evaluated for an arbitrary stationary probability measure or for a universal code. We call these rates Hilberg exponents, to commemorate Hilberg's insight. The formal definition rests on the following preliminaries:

- (i) Let \mathbb{X} be a countable alphabet. Consider a probability space (Ω, \mathcal{J}, Q) with $\Omega = \mathbb{X}^{\mathbb{Z}}$, discrete random variables $X_k : \Omega \ni (x_i)_{i \in \mathbb{Z}} \mapsto x_k \in \mathbb{X}$, and a probability measure Q which is stationary on $(X_i)_{i \in \mathbb{Z}}$ but not necessarily ergodic. Blocks of symbols or variables are denoted as $X_n^m = (X_i)_{n \leq i \leq m}$. We introduce shorthand notation $Q(x_1^n) = Q(X_1^n = x_1^n)$. The expectation of random variable X with respect to Q is written $\mathbf{E}_Q X$ and the variance is $\text{Var}_Q X$.
- (ii) Moreover, measure Q will be compared with codes, which for uniformity of notation will be represented in our approach as incomplete measures P , i.e., a code P in our approach is a function that satisfies $P(x_1^n) \geq 0$ and the Kraft inequality $\sum_{x_1^n} P(x_1^n) \leq 1$.¹ Subsequently, for a code P (or for measure Q), we define the pointwise mutual information between blocks

$$I^P(n) = -\log P(X_{-n+1}^0) - \log P(X_1^n) + \log P(X_{-n+1}^n). \quad (1)$$

In the formula \log stands for the binary logarithm.

Now we may define the Hilberg exponents:

Definition 1 (Hilberg exponents) *Define the positive logarithm*

$$\log^+ x = \begin{cases} \log(x+1), & x \geq 0, \\ 0, & x < 0. \end{cases} \quad (2)$$

¹Here we deviate from the standard definition. Usually a code is a function $C : \mathbb{X}^* \rightarrow \{0, 1\}^*$ with length $|C(x_1^n)|$, where $2^{-|C(x_1^n)|}$ is a certain code in our sense. An instance of such a code C is the Lempel-Ziv code [11].

For a code P we introduce

$$\gamma_P^+ = \limsup_{n \rightarrow \infty} \frac{\log^+ I^P(n)}{\log n}, \quad (3)$$

$$\gamma_P^- = \liminf_{n \rightarrow \infty} \frac{\log^+ I^P(n)}{\log n}, \quad (4)$$

$$\delta_P^+ = \limsup_{n \rightarrow \infty} \frac{\log^+ \mathbf{E}_Q I^P(n)}{\log n}, \quad (5)$$

$$\delta_P^- = \liminf_{n \rightarrow \infty} \frac{\log^+ \mathbf{E}_Q I^P(n)}{\log n}. \quad (6)$$

The above numbers will be called: γ_P^+ —the upper random Hilberg exponent, γ_P^- —the lower random Hilberg exponent, δ_P^+ —the upper expected Hilberg exponent, and δ_P^- —the lower expected Hilberg exponent.

Exponents γ_P^\pm are random variables, whereas δ_P^\pm are constants. By definition,

$$\gamma_P^+ \geq \gamma_P^- \geq 0, \quad (7)$$

$$\delta_P^+ \geq \delta_P^- \geq 0. \quad (8)$$

Let us remark that Hilberg exponents for $P = Q$ quantify some sort of long-range non-Markovian dependence in the process. In particular, for Q being the measure of an IID process or a hidden Markov process with a finite number of hidden states, mutual information $\mathbf{E}_Q I^Q(n)$ is zero or bounded, respectively, and hence $\delta_Q^\pm = \gamma_Q^\pm = 0$. The same is true for k -parameter sources since $I^Q(n)$ is proportional to $k \log n$ [3, 4, 5]. However, if information $I^Q(n)$ grows proportionally to n^β where $\beta \in [0, 1]$ then $\gamma_Q^\pm = \delta_Q^\pm = \beta$. There exist some simple non-Markovian but still mixing sources [7], being a generalization of the Santa Fe processes, for which δ_Q^\pm can be an arbitrary number in range $(0, 1)$.

There are a few reasons why we introduce so many Hilberg exponents, for each stationary measure Q and each code P . The first one is that the pointwise mutual information $I^P(n)$ may grow by leaps and bounds. Consequently, the upper bounding power-law function may rise faster than the respective lower bounding power-law function. This may happen indeed. The second reason is that, a priori, different rates of growth might be observed for the pointwise and the expected mutual information. If they are equal, this should be separately proved. As for the final reason, whereas we are here most interested in case $P = Q$, some reason for investigating the pointwise mutual information $I^P(n)$ for $P \neq Q$ is that, paradoxically, sometimes it is easier to say something about Q -typical behavior of $I^P(n)$ than about $I^Q(n)$. As we will show, this concerns not only statistical applications, where we do not know Q , but also some theoretical results, where Q is known. Thus our definition is not too generic.

In the following we will show that Hilberg exponents satisfy a number of relationships that impose some order among them. The first thing we note are inequalities

$$\gamma_P^- \leq \gamma_P^+ \leq 1, \quad (9)$$

$$\delta_P^- \leq \delta_P^+ \leq 1, \quad (10)$$

which hold in the following cases:

- (i) For $P = Q$: Let $k(n)$ and $l(n)$ be nondecreasing functions of n , where $k(n) + l(n) \rightarrow \infty$. By an easy generalization of the Shannon-McMillan-Breiman theorem [14, 15, 16], we have Q -almost surely that

$$\lim_{n \rightarrow \infty} \frac{1}{k(n) + l(n) + 1} \left[-\log Q(X_{-k(n)}^{l(n)}) \right] = h_Q, \quad (11)$$

where h_Q is the entropy rate of measure Q (h_Q is a random variable if Q is nonergodic). Hence $\lim_{n \rightarrow \infty} I^Q(n)/n = 0$ and so $\gamma_Q^- \leq \gamma_Q^+ \leq 1$ holds Q -almost surely. Moreover, by stationarity,

$$\lim_{n \rightarrow \infty} \frac{1}{k(n) + l(n) + 1} \mathbf{E}_Q \left[-\log Q(X_{-k(n)}^{l(n)}) \right] = \mathbf{E}_Q h_Q. \quad (12)$$

Hence $\lim_{n \rightarrow \infty} \mathbf{E}_Q I^Q(n)/n = 0$ and so $\delta_Q^- \leq \delta_Q^+ \leq 1$.

- (ii) For P being universal almost surely and in expectation: Let $k(n)$ and $l(n)$ be nondecreasing functions of n , where $k(n) + l(n) \rightarrow \infty$, as previously. Here, we will say that a code P is universal almost surely if for every stationary distribution Q we have Q -almost surely

$$\lim_{n \rightarrow \infty} \frac{1}{k(n) + l(n) + 1} \left[-\log P(X_{-k(n)}^{l(n)}) \right] = h_Q, \quad (13)$$

where h_Q is the entropy rate of measure Q . In that case $\lim_{n \rightarrow \infty} I^P(n)/n = 0$ so $\gamma_P^- \leq \gamma_P^+ \leq 1$ holds Q -almost surely. Moreover, we will say that a code P is universal in expectation if for every stationary distribution Q we have

$$\lim_{n \rightarrow \infty} \frac{1}{k(n) + l(n) + 1} \mathbf{E}_Q \left[-\log P(X_{-k(n)}^{l(n)}) \right] = \mathbf{E}_Q h_Q. \quad (14)$$

In that case $\lim_{n \rightarrow \infty} \mathbf{E}_Q I^P(n)/n = 0$ so $\delta_P^- \leq \delta_P^+ \leq 1$.

Equality (13) can be satisfied since stationary ergodic measures are mutually singular. Moreover, almost surely universal codes exist if and only if the alphabet \mathbb{X} is finite [17]. Some example of an almost surely universal code is the Lempel-Ziv code [11]. We also note that an almost surely universal code P is universal in expectation if $-\log P(X_{-k(n)}^{l(n)}) \leq C(k(n) + l(n) + 1)$ for some constant $C > 0$ [18]. In particular, the Lempel-Ziv code satisfies this inequality.

As we have indicated, there are quite many Hilberg exponents, for different measures and for different codes. Seeking for some order in this menagerie, we may look for results of three kinds:

- (i) For a fixed code P and a measure Q , we relate the random exponents γ_P^\pm and the expected exponents δ_P^\pm .
- (ii) For two codes P and R , we relate the exponents of a fixed kind, say δ_P^\pm and δ_R^\pm for some measure Q .
- (iii) For a fixed code P and a measure Q , we directly evaluate exponents γ_P^\pm and δ_P^\pm .

In the following we will present some results of these three sorts. They have varying weight but they shed some light onto unknown territory.

The first kind of results could be called “second-order” analogues of the Shannon-McMillan-Breiman (SMB) theorem (11). The original idea of the SMB theorem was to relate the asymptotic growth of pointwise and expected entropies for an ergodic process Q with $P = Q$. With some partial success, this idea was then extended to the case when the code P was different to the underlying measure Q [19, 20, 21]. In contrast, relating the random Hilberg exponents γ_P^\pm and the expected Hilberg exponents δ_P^\pm means relating the speed of growth of the pointwise and expected mutual informations, which are differences of the respective entropies. This is a somewhat subtler effect than the SMB theorem, hence our “second-order” terminology. In this domain we have achieved an interesting result. For an arbitrary code P with exponent $\delta_P^- > 0$, let us introduce parameter

$$\epsilon_P = \limsup_{n \rightarrow \infty} \frac{\log^+ [\text{Var}_Q I^P(n) / \mathbf{E}_Q I^P(n)]}{\log n}. \quad (15)$$

Our result is a sandwich bound for random Hilberg exponents in terms of the expected Hilberg exponents for $P = Q$:

Theorem 1 *For an ergodic measure Q over a finite alphabet, random Hilberg exponents γ_Q^\pm are almost surely constant. Moreover, we have Q -almost surely*

$$\delta_Q^+ \geq \gamma_Q^+ \geq \delta_Q^+ - \epsilon_Q, \quad (16)$$

$$\delta_Q^- \geq \gamma_Q^- \geq \delta_Q^- - \epsilon_Q, \quad (17)$$

where the left inequalities hold without restrictions, whereas the right inequalities hold for $\delta_Q^- > 0$.

As we have written, this theorem may be considered an analogue of the SMB theorem for the mutual information of the underlying measure.

We cannot refrain from mentioning the uncommon technique that has led us to proving Theorem 1. It is remarkable that this result can be demonstrated without invoking the ergodic theorem. Instead, we use an auxiliary “Kolmogorov code”

$$S(x_1^n) = 2^{-K(x_1^n|F)}, \quad (18)$$

where $K(x_1^n|F)$ is the prefix-free Kolmogorov complexity of a string x_1^n given an object F on an additional infinite tape [22, 23]. The object F can be another string or, in our application, a definition of some measure. Respectively, quantity $I^S(n)$ is the conditional algorithmic mutual information. By some approximate translation invariance of Kolmogorov complexity, we can show that the random Hilberg exponents γ_S^\pm are almost surely constant on ergodic sources Q . This fact constitutes a novel contribution of algorithmic information theory to the study of stochastic processes. Further, using Markov inequality and Borel-Cantelli lemma, we can show that $\gamma_S^\pm \leq \delta_S^\pm$ on ergodic sources Q , as well. To complete the rough idea of the proof of Theorem 1, let us mention that $\gamma_S^\pm = \gamma_Q^\pm$ and $\delta_S^\pm = \delta_Q^\pm$ if we condition the Kolmogorov complexity on the distribution Q , i.e., if we plug $F = Q$ in (18). This follows by Shannon-Fano

coding and Barron inequality [24, Theorem 3.1]. In this way we obtain the left inequalities in (16)–(17). The right inequalities are demonstrated in quite a similar fashion, using some auxiliary quantities for the Kolmogorov code S , which we will call inverse Hilberg exponents.

Now let us proceed to the second kind of results, those for two codes. Here our results are modest. Let us note that in applications we often do not know the underlying measure and we cannot compute the Kolmogorov complexity but we can compute some other universal codes such as the Lempel-Ziv code. Thus it would be advisable to relate Hilberg exponents for computable (in the sense of the theory of computation) universal codes to Hilberg exponents for the Kolmogorov code S or the underlying measure Q . For a universal code, we may suppose that the longer the code is, the larger Hilberg exponents it has. This hope is partly confirmed by the following simple theorem.

Theorem 2 *Let f_n be such that*

$$\limsup_{n \rightarrow \infty} \frac{\log^+ |f_n|}{\log n} = 0. \quad (19)$$

Suppose that for codes P and R and a stationary measure Q we have

$$\mathbf{E}_Q [-\log P(X_1^n)] \leq \mathbf{E}_Q [-\log R(X_1^n)] + f_n, \quad (20)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbf{E}_Q [-\log P(X_1^n)] = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbf{E}_Q [-\log R(X_1^n)]. \quad (21)$$

Then

$$\delta_R^+ \geq \delta_P^-. \quad (22)$$

The simple proof of the above proposition rests on this lemma:

Lemma 1 ([6]) *Consider a function $G : \mathbb{N} \rightarrow \mathbb{R}$ such that $\lim_k G(k)/k = 0$ and $G(n) \geq 0$ for all but finitely many n . For infinitely many n , we have $2G(n) - G(2n) \geq 0$.*

To prove Theorem 2, it suffices to put $G(n) = -\log R(X_1^n) + \log P(X_1^n) + f_n$ and use subadditivity of the function \log^+ , i.e., inequality

$$\log^+(x + y) \leq \log^+ x + \log^+ y. \quad (23)$$

The most useful applications of Theorem 2 are as follows: Condition (20) is satisfied, with $f_n = 2 \log n + C$, where $C > 0$, for any computable code R and unconditional Kolmogorov code $P(x_1^n) = 2^{-K(x_1^n)}$, where $K(x_1^n)$ is the unconditional prefix-free Kolmogorov complexity of a string x_1^n . Moreover condition (20) is satisfied, with $f_n = 0$, for any code R and $P = Q$. Condition (21) is satisfied if P and R are universal in expectation or if R is universal in expectation and $P = Q$. Moreover, conditions (20) and (21) are satisfied with $f_n = 0$ for $R = Q$ and $P = E$ [25], where E is the random ergodic measure given by

$$E = Q(\cdot | \mathcal{I}) \quad (24)$$

where \mathcal{I} is the shift-invariant algebra [26, 6].

Let us remark that inequality (22) is not very strong. We have been able to relate only the upper expected Hilberg exponents with the lower expected

Hilberg exponent. It would be more interesting if, for two different codes, we were able to compare the random exponents of the same kind, i.e., an upper exponent with an upper exponent and a lower exponent with a lower exponent. This requires relating functions $I^P(n)$ and $I^R(n)$. But even relatively simple cases, such as comparing $I^Q(n)$ and $I^E(n)$, where E is the random ergodic measure (24), are not trivial. In that case, $I^Q(n) - I^E(n)$ equals triple mutual information between two blocks and the shift-invariant algebra, which may be negative. Therefore we leave strengthening Theorem 2 as an open problem.

To end this introduction, we mention the third kind of results, which concern analytic evaluation of Hilberg exponents for concrete examples of processes. As we have indicated, the expected Hilberg exponents for IID processes, Markov processes, and hidden Markov processes do vanish because the mutual information is either zero or bounded. To investigate less trivial cases, in this paper, we will consider three kinds of other sources. These will be: the mixture Bernoulli process, the original Santa Fe processes mentioned in the very beginning of this paper [6, 7], and some modification of the Santa Fe processes. It should be noted that all analyzed processes are conditionally IID. The pointwise mutual information $I^Q(n)$ for such sources is equal to a difference of redundancies. As shown in [3, 4, 5], in case of k -parameter processes, such as the mixture Bernoulli process, the redundancy grows proportionally to $k \log n$. For this reason all Hilberg exponents do vanish for the mixture Bernoulli process. For completeness we will reproduce the relevant simple calculation in this paper. In contrast, the second example, the original Santa Fe process, exhibits a stronger dependence, namely $I^Q(n)$ grows proportionally to n^β , where $\beta \in (0, 1)$ is a certain free parameter of the process. Therefore all four Hilberg exponents are equal to β . In the third example, we can, however, modify the definition of the Santa Fe process so that the upper expected exponent is $\delta_Q^+ = \beta$ for an arbitrary parameter $\beta \in (0, 1)$ whereas the lower expected exponent is $\delta_Q^- = 0$. This shows that upper and lower Hilberg exponents need not be equal.

The further contents of the paper is as follows. In Section II, we discuss properties of Hilberg exponents for the Kolmogorov code S . In Section III, we translate these results for the underlying measure Q , proving Theorem 1. Finally, in Section IV, we evaluate Hilberg exponents for the mixture Bernoulli process and the Santa Fe processes.

II Kolmogorov code

A prominent role in our demonstration of Theorem 1 will be played by the Kolmogorov code (18), whose properties will be discussed in this section. Although the results of this section are used in the next section to prove Theorem 1, they can be regarded as facts of some independent interest. For this reason, we assemble them into a separate narrative unit.

To begin with, let us note that, for a finite alphabet \mathbb{X} , the Kolmogorov code is universal almost surely and in expectation, simply because it is dominated by the Lempel-Ziv code. Independently, universality of the Kolmogorov code has been previously shown by Brudno [27] in the context of dynamical systems. Although Kolmogorov complexity itself is uncomputable, the Hilberg exponents for the Kolmogorov code can be evaluated in some cases and enjoy a few nice properties. These properties stem from the fact that function $I^S(n)$ equals the

algorithmic mutual information

$$I^S(n) = K(X_{-n+1}^0|F) + K(X_1^n|F) - K(X_{-n+1}^n|F), \quad (25)$$

an important concept in algorithmic information theory.

Now let us present some new results. A simple but important fact is that the random Hilberg exponents are almost surely constant on ergodic sources. This fact is a consequence of approximate shift-invariance of Kolmogorov complexity. That property seemingly has not been noticed so far and it provides an interesting link between algorithmic complexity and ergodic theory.

Theorem 3 *Consider code (18) and an ergodic measure Q over a finite alphabet \mathbb{X} . Exponents γ_S^- and γ_S^+ are Q -almost surely constant.*

Remark: The random Hilberg exponents can be different for different ergodic sources, so for nonergodic sources they can be random.

Proof: For $t > 0$, from the shortest program that computes x_1^n , we can construct a program that computes x_{t+1}^{t+n} , whose length exceeds the length of the program for x_1^n no more than $K(x_{n+1}^{n+t}|F) + C$, where $C > 0$. Analogously, from the shortest program that computes x_{t+1}^{t+n} , we can construct a program that computes x_1^n , whose length exceeds the length of the program for x_{t+1}^{t+n} no more than $K(x_1^t|F) + C$. This yields

$$|K(x_1^n|F) - K(x_{t+1}^{t+n}|F)| \leq \max \{K(x_1^t|F), K(x_{n+1}^{n+t}|F)\} + C.$$

Thus

$$\begin{aligned} |I^S(n) - [K(X_{t-n+1}^t|F) - K(X_{t+1}^{t+n}|F) + K(X_{t-n+1}^{t+n}|F)]| \\ \leq 3 \max \{K(X_{-n+1}^{-n+t}|F), K(X_1^t|F), K(X_{n+1}^{n+t}|F)\} + 3C. \end{aligned}$$

Now we notice that for a finite alphabet we have

$$K(X_{-n+1}^{-n+t}|F), K(X_1^t|F), K(X_{n+1}^{n+t}|F) \leq Ct,$$

where $C > 0$. Hence by inequality (23) functions γ_S^- and γ_S^+ are shift-invariant. Since Q is ergodic, it means they must be Q -almost surely constant. \square

Subsequently, we will give some bounds for the random Hilberg exponents in terms of the expected Hilberg exponents. To achieve this goal, we need two lemmas and some additional definition. In the following, we will write $a(n) \overset{+}{>} b(n)$ if $a(n) + C \geq b(n)$ for all arguments n and a $C \geq 0$, whereas $a(n) \overset{+}{<} b(n)$ if $a(n) \leq b(n) + C$ under the same conditions. We also write $a(n) \overset{\pm}{>} b(n)$ if $a(n) \overset{+}{>} b(n)$ and $a(n) \overset{+}{<} b(n)$.

The following lemma is the first step on our way. It says that mutual information $I^S(n)$ is almost a nondecreasing function.

Lemma 2 *Consider code (18). For all $m \geq 1$, we have*

$$I^S(n+m) \overset{+}{>} I^S(n) - 4 \log m. \quad (26)$$

Proof: For strings u and v , denote the algorithmic mutual information

$$I(u : v|F) = K(u|F) + K(v|F) - K(u, v|F). \quad (27)$$

We have $I(u : v|F) \stackrel{+}{>} 0$ [23]. Concatenating strings decreases their complexity. Namely,

$$K(uv|F) \stackrel{+}{<} K(u, v|F) \stackrel{+}{<} K(uv|F) + K(|v| | uv, F),$$

where $|v|$ is the length of v . Hence

$$I^S(n + m) + 4 \log m \stackrel{+}{>} I(a, b : c, d|F)$$

whereas

$$I^S(n) \stackrel{\pm}{=} I(b : c|F)$$

where $a = X_{-n+m+1}^{-n}$, $b = X_{-n+1}^0$, $c = X_1^n$, and $d = X_{n+1}^m$.

Using identity $K(u, v|F) \stackrel{\pm}{=} K(u|F) + K(v|u, K(u|F), F)$ [23], we can further show the data processing inequality for the algorithmic mutual information,

$$\begin{aligned} I(a, b : c, d|F) &\stackrel{\pm}{=} I(a, b : c|F) + I(a, b : d|c, K(c|F), F) \\ &\stackrel{+}{>} I(a, b : c|F) \\ &\stackrel{\pm}{=} I(b : c|F) + I(a : c|b, K(b|F), F) \\ &\stackrel{+}{>} I(b : c|F), \end{aligned}$$

which proves the claim. \square

With the above lemma we can prove another auxiliary result. This result says that Hilberg exponents for the Kolmogorov code can be defined using only a subsequence of exponentially growing block lengths.

Lemma 3 *Consider code (18). We have*

$$\gamma_S^+ = \limsup_{k \rightarrow \infty} \frac{\log^+ I^S(2^k)}{\log 2^k}, \quad (28)$$

$$\gamma_S^- = \liminf_{k \rightarrow \infty} \frac{\log^+ I^S(2^k)}{\log 2^k}, \quad (29)$$

$$\delta_S^+ = \limsup_{k \rightarrow \infty} \frac{\log^+ \mathbf{E}_Q I^S(2^k)}{\log 2^k}, \quad (30)$$

$$\delta_S^- = \liminf_{k \rightarrow \infty} \frac{\log^+ \mathbf{E}_Q I^S(2^k)}{\log 2^k}. \quad (31)$$

Proof: By Lemma 2, for $n = 2^k + m$, where $0 \leq m < 2^k$, we have

$$I^S(n) \leq I^S(2^{k+1}) + 4 \log(2^k - m) + C \leq I^S(2^{k+1}) + 4k + C.$$

Thus

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \frac{\log^+ I^S(n)}{\log n} &\leq \limsup_{k \rightarrow \infty} \frac{\log^+(I^S(2^{k+1}) + 4k + C)}{\log 2^k} \\
&\leq \limsup_{k \rightarrow \infty} \frac{\log^+ I^S(2^{k+1})}{\log 2^k} + \limsup_{k \rightarrow \infty} \frac{\log^+(4k + C)}{\log 2^k} \\
&= \limsup_{k \rightarrow \infty} \frac{\log^+ I^S(2^k)}{\log 2^k}.
\end{aligned}$$

This proves (28) since trivially we have a converse inequality.

Now let $n = 2^k + m$, where $0 < m \leq 2^k$. From (26), we obtain

$$I^S(n) \geq I^S(2^k) - 4 \log m - C \geq I^S(2^k) - 4k - C.$$

Thus

$$\begin{aligned}
\liminf_{n \rightarrow \infty} \frac{\log^+ I^S(n)}{\log n} &\geq \liminf_{k \rightarrow \infty} \frac{\log^+(I^S(2^k) - 4k - C)}{\log 2^{k+1}} \\
&\geq \liminf_{k \rightarrow \infty} \frac{\log^+ I^S(2^k)}{\log 2^{k+1}} - \limsup_{k \rightarrow \infty} \frac{\log^+(4k + C)}{\log 2^{k+1}} \\
&= \liminf_{k \rightarrow \infty} \frac{\log^+ I^S(2^k)}{\log 2^k}.
\end{aligned}$$

This proves (29) for trivially we have a converse inequality.

The proofs of (30) and (31) are analogous. \square

To finish the preparations, we need some auxiliary concept. Recall that algorithmic mutual information (27) is greater than a constant. Using this result, for the Kolmogorov code we can introduce the following inverse Hilberg exponents.

Definition 2 (inverse Hilberg exponents) Consider code (18). Let B be such that $I^S(n) + B \geq 1$. Define

$$\zeta_S^+ = \limsup_{n \rightarrow \infty} \frac{\log^+ [\mathbf{E}_Q(I^S(n) + B)^{-1}]^{-1}}{\log n}, \quad (32)$$

$$\zeta_S^- = \liminf_{n \rightarrow \infty} \frac{\log^+ [\mathbf{E}_Q(I^S(n) + B)^{-1}]^{-1}}{\log n}. \quad (33)$$

The above numbers will be called: ζ_S^+ —the upper inverse expected Hilberg exponent and ζ_S^- —the lower inverse expected Hilberg exponent.

We have $\zeta_S^+ \geq \zeta_S^- \geq 0$, whereas $\delta_S^+ \geq \zeta_S^+$ and $\delta_S^- \geq \zeta_S^-$ by the Jensen inequality $\mathbf{E}_Q X \geq [\mathbf{E}_Q X^{-1}]^{-1}$ for $X > 0$.

Now we may state and prove the theorem which links the expected and the random Hilberg exponents for the Kolmogorov code. It will be first stated and proved for general stationary (not necessarily ergodic) measures over a countable alphabet. Subsequently, we will present a corollary for ergodic measures and some strengthening for a finite alphabet.

Theorem 4 Consider code (18) and an arbitrary stationary measure Q . Then:

(i) $\delta_S^+ \geq \gamma_S^+$ Q -almost surely and $\text{ess sup}_Q \gamma_S^+ \geq \zeta_S^+$.

(ii) $\delta_S^- \geq \text{ess inf}_Q \gamma_S^-$ and $\gamma_S^- \geq \zeta_S^-$ Q -almost surely.

Proof:

(i) Let $\epsilon > 0$. Observe that from the Markov inequality we have

$$\begin{aligned} \sum_{k=1}^{\infty} Q \left(\frac{I^S(2^k) + B}{(2^k)^{\delta_S^+ + \epsilon}} \geq 1 \right) &\leq \sum_{k=1}^{\infty} \frac{\mathbf{E}_Q I^S(2^k) + B}{(2^k)^{\delta_S^+ + \epsilon}} \\ &\leq A + \sum_{k=1}^{\infty} \frac{(2^k)^{\delta_S^+ + \epsilon/2}}{(2^k)^{\delta_S^+ + \epsilon}} < \infty, \end{aligned}$$

where $A < \infty$. Hence, by the Borel-Cantelli lemma, we have Q -almost surely

$$\limsup_{k \rightarrow \infty} \frac{\log^+(I^S(2^k) + B)}{\log 2^k} \leq \delta_S^+ + \epsilon.$$

By arbitrariness of ϵ and by inequality (23), the bound is true with $\epsilon = 0$ and $B = 0$, which implies $\delta_S^+ \geq \gamma_S^+$ Q -almost surely by Lemma 3.

Now we will prove that $\text{ess sup}_Q \gamma_S^+ \geq \zeta_S^+$. Denote $\text{ess sup}_Q \gamma_S^+ = \beta$ and let $\epsilon > 0$. Then $Q(\gamma_S^+ > \beta + \epsilon/2) = 0$ whence

$$Q(I^S(n) + B \geq n^{\beta + \epsilon} \text{ infinitely often}) = 0. \quad (34)$$

Denote $p(n) = Q(I^S(n) + B < n^{\beta + \epsilon})$. We have

$$\mathbf{E}_Q (I^S(n) + B)^{-1} \geq n^{-\beta - \epsilon} p(n).$$

By (34), $\lim_{n \rightarrow \infty} p(n) = 1$. Hence $\zeta_S^+ \leq \beta + \epsilon$. Since ϵ was arbitrary, this implies the claim.

(ii) The proof is analogous to the proof of (i). Write $\text{ess inf}_Q \gamma_S^- = \beta$ and let $\epsilon > 0$. Then $Q(\gamma_S^- < \beta - \epsilon/2) = 0$ whence

$$Q(I^S(n) \leq n^{\beta - \epsilon} \text{ infinitely often}) = 0. \quad (35)$$

Denote $p(n) = Q(I^S(n) > n^{\beta - \epsilon})$. We have

$$\mathbf{E}_Q I^S(n) \geq n^{\beta - \epsilon} p(n).$$

By (35), $\lim_{n \rightarrow \infty} p(n) = 1$. Thus $\delta_S^- \geq \beta - \epsilon$. Since ϵ was arbitrary, this implies $\delta_S^- \geq \text{ess inf}_Q \gamma_S^-$.

Now we will show the second claim. Let $\epsilon > 0$. From the Markov inequality

$$\begin{aligned} \sum_{k=1}^{\infty} Q \left(\frac{I^S(2^k) + B}{(2^k)^{\zeta_S^- - \epsilon}} \leq 1 \right) &\leq \sum_{k=1}^{\infty} \frac{\mathbf{E}_Q (I^S(2^k) + B)^{-1}}{(2^k)^{-\zeta_S^- + \epsilon}} \\ &\leq A + \sum_{k=1}^{\infty} \frac{(2^k)^{-\zeta_S^- + \epsilon/2}}{(2^k)^{-\zeta_S^- + \epsilon}} < \infty, \end{aligned}$$

where $A < \infty$. Thus, by the Borel-Cantelli lemma, Q -almost surely

$$\liminf_{k \rightarrow \infty} \frac{\log^+(I^S(2^k) + B)}{\log 2^k} \geq \zeta_S^- - \epsilon.$$

As in (i), we may put $\epsilon = 0$ and $B = 0$, whence $\gamma_S^- \geq \zeta_S^-$ Q -almost surely follows by Lemma 3.

□

Let us also present some add-ons to Theorem 4. Using Theorem 3, Theorem 4 can be specialized for ergodic measures over a finite alphabet in an interesting way, which will be used later.

Corollary 1 *By Theorem 3, for an ergodic measure Q over a finite alphabet, equalities $\gamma_S^+ = \text{ess sup}_Q \gamma_S^+$ and $\gamma_S^- = \text{ess inf}_Q \gamma_S^-$ hold Q -almost surely. Hence, Q -almost surely we have*

$$\delta_S^+ \geq \gamma_S^+ \geq \zeta_S^+, \quad (36)$$

$$\delta_S^- \geq \gamma_S^- \geq \zeta_S^-. \quad (37)$$

It is remarkable that inequalities (36) and (37) are demonstrated without invoking the ergodic theorem.

Let us observe one more simple fact. Namely, for a finite alphabet \mathbb{X} , the bound for the random Hilberg exponents given by Theorem 4 can be slightly strengthened since $\text{ess sup}_Q \gamma_S^+ \geq \mathbf{E}_Q \gamma_S^+$ and $\mathbf{E}_Q \gamma_S^- \geq \text{ess inf}_Q \gamma_S^-$.

Theorem 5 *Consider code (18) and an arbitrary stationary measure Q . Then:*

(i) $\mathbf{E}_Q \gamma_S^+ \geq \zeta_S^+$ if the alphabet \mathbb{X} is finite.

(ii) $\delta_S^- \geq \mathbf{E}_Q \gamma_S^-$.

Proof:

(i) Function $-\log$ is convex. Hence we can use the Fatou lemma and the Jensen inequality,

$$\begin{aligned} \mathbf{E}_Q \gamma_S^+ &= \mathbf{E}_Q \limsup_{n \rightarrow \infty} \frac{\log(I^S(n) + B)}{\log n} \\ &\geq \limsup_{n \rightarrow \infty} \mathbf{E}_Q \frac{\log(I^S(n) + B)}{\log n} \\ &\geq \limsup_{n \rightarrow \infty} \mathbf{E}_Q \left[-\frac{\log(I^S(n) + B)^{-1}}{\log n} \right] \\ &\geq \limsup_{n \rightarrow \infty} \left[-\frac{\log \mathbf{E}_Q(I^S(n) + B)^{-1}}{\log n} \right] = \zeta_S^+, \end{aligned}$$

since the functions under the limits are bounded above.

(ii) Reasoning as above,

$$\begin{aligned}
\mathbf{E}_Q \gamma_S^- &= \mathbf{E}_Q \liminf_{n \rightarrow \infty} \frac{\log(I^S(n) + B)}{\log n} \\
&\leq \liminf_{n \rightarrow \infty} \mathbf{E}_Q \frac{\log(I^S(n) + B)}{\log n} \\
&\leq \liminf_{n \rightarrow \infty} \frac{\log \mathbf{E}_Q(I^S(n) + B)}{\log n} = \delta_S^-,
\end{aligned}$$

since the functions under the limits are nonnegative.

□

III The underlying measure

In this section we will prove Theorem 1, which provides a bound for the random Hilberg exponents of the underlying measure Q in terms of the measure's expected Hilberg exponents. For this goal, we will use the results of the previous section. Our technique rests on a few observations. The first observation is that four out of six Hilberg exponents for the Kolmogorov code are equal to the Hilberg exponents for the underlying measure Q if we use a special conditional Kolmogorov code. In this code, the definition of measure Q is fed to the Turing machine on an additional infinite tape, i.e., $F = Q$. By the Shannon-Fano coding and the Barron inequality, such a Kolmogorov code is equal to the measure Q in a sufficiently good approximation.

Theorem 6 *Consider code (18) with $F = Q$, where Q is an arbitrary stationary measure. Then:*

- (i) $\delta_S^- = \delta_Q^-$ and $\delta_S^+ = \delta_Q^+$.
- (ii) $\gamma_S^- = \gamma_Q^-$ and $\gamma_S^+ = \gamma_Q^+$ Q -almost surely.

Proof:

- (i) The Shannon-Fano coding gives

$$-\log S(x_1^n) = K(x_1^n | Q) \leq -\log Q(x_1^n) + 2 \log n + C \quad (38)$$

for a constant $C > 0$ [28]. Hence from the source coding inequality

$$\mathbf{E}_Q H^S(n) \geq \mathbf{E}_Q H^Q(n),$$

we obtain

$$|\mathbf{E}_Q I^S(n) - \mathbf{E}_Q I^Q(n)| \leq 4 \log n + 2C. \quad (39)$$

Thus by inequality (23), $\delta_S^- = \delta_Q^-$ and $\delta_S^+ = \delta_Q^+$.

- (ii) Observe that Q -almost surely we have the following Barron inequalities, viz. [24, Theorem 3.1],

$$\begin{aligned}\lim_{n \rightarrow \infty} [-\log S(X_{-n+1}^n) + \log Q(X_{-n+1}^n)] &= \infty, \\ \lim_{n \rightarrow \infty} [-\log S(X_1^n) + \log Q(X_1^n)] &= \infty, \\ \lim_{n \rightarrow \infty} [-\log S(X_{-n+1}^{2n}) + \log Q(X_{-n+1}^n)] &= \infty.\end{aligned}$$

Combining these facts with the Shannon-Fano coding (38) yields

$$|I^S(n) - I^Q(n)| \leq 4 \log n + 2C.$$

for sufficiently large n , Q -almost surely. Thus by inequality (23), $\gamma_S^- = \gamma_Q^-$ and $\gamma_S^+ = \gamma_Q^+$ holds on a set of full measure.

□

Theorem 6 implies two more specific corollaries of an independent interest. The first result states that Hilberg exponents for a computable measure Q are equal to Hilberg exponents for unconditional prefix-free Kolmogorov complexity.

Corollary 2 *If measure Q is computable then for code $P(x_1^n) = 2^{-K(x_1^n)}$, where $K(x_1^n)$ is unconditional prefix-free Kolmogorov complexity of x_1^n , we have*

$$I^P(n) \stackrel{\pm}{=} I^S(n),$$

where we use code (18) with $F = Q$ again. This implies $\gamma_Q^- = \gamma_S^- = \gamma_P^-$ and $\gamma_Q^+ = \gamma_S^+ = \gamma_P^+$ Q -almost surely, whereas $\delta_Q^- = \delta_S^- = \delta_P^-$ and $\delta_Q^+ = \delta_S^+ = \delta_P^+$.

The second result concerns a nonergodic measure with a given ergodic decomposition. It says that Hilberg exponents for this nonergodic measure are almost surely constant on almost all ergodic components of the measure.

Corollary 3 *Suppose that measure Q has the random ergodic measure E given by (24). We have $Q = \mathbf{E}_Q E$, so by the properties of integral, any set of full Q -measure has full E -measure Q -almost surely. This implies that for code (18) with $F = Q$, we have $\gamma_S^- = \gamma_Q^-$ and $\gamma_S^+ = \gamma_Q^+$ E -almost surely for Q -almost all values of measure E . By Theorem 3, in case of a finite alphabet, this means that γ_Q^- and γ_Q^+ are E -almost surely constant for those values of measure E .*

What lacks for the proof of Theorem 1 is a computable lower bound for the inverse Hilberg exponents, defined in the previous section for the Kolmogorov code S . For an arbitrary code P with $\delta_P^- > 0$, let us introduce parameter ϵ_P given by formula (15). First, we will show that the difference between the expected and the inverse Hilberg exponents $\delta_S^\pm - \zeta_S^\pm$ is bounded by parameter ϵ_S and then we will show that $\epsilon_S = \epsilon_Q$ for $F = Q$.

Theorem 7 *Consider code (18) and an arbitrary stationary measure Q . If $\delta_S^- > 0$ then $\zeta_S^+ \geq \delta_S^+ - \epsilon_S$ and $\zeta_S^- \geq \delta_S^- - \epsilon_S$.*

Proof: Let $\alpha \in (0, 1)$. By $I^S(n) + B \geq 1$ and by Markov inequality we obtain

$$\begin{aligned} \mathbf{E}_Q(I^S(n) + B)^{-1} &\leq Q(|I^S(n) - \mathbf{E}_Q I^S(n)| \geq \alpha(\mathbf{E}_Q I^S(n) + B)) \\ &\quad + \frac{1}{(1 - \alpha)(\mathbf{E}_Q I^S(n) + B)} \\ &\leq \frac{\text{Var}_Q I^S(n)}{\alpha^2(\mathbf{E}_Q I^S(n) + B)^2} + \frac{1}{(1 - \alpha)(\mathbf{E}_Q I^S(n) + B)}. \end{aligned}$$

Hence

$$[\mathbf{E}_Q(I^S(n) + B)^{-1}]^{-1} \geq (\mathbf{E}_Q I^S(n) + B) \left(\frac{\text{Var}_Q I^S(n)}{\alpha^2(\mathbf{E}_Q I^S(n) + B)} + \frac{1}{(1 - \alpha)} \right)^{-1},$$

which implies the claim by $\log(x/y) = \log x - \log y$, $\delta_S^- > 0$, and inequality (23). \square

Subsequently, we will prove that parameter ϵ_S for the conditional Kolmogorov code with $F = Q$ is equal to parameter ϵ_Q for the underlying measure.

Theorem 8 *Consider code (18) with $F = Q$, where Q is an arbitrary stationary measure. If $\delta_Q^- > 0$ then $\epsilon_S = \epsilon_Q$.*

Proof: Since $\delta_Q^- > 0$, by inequality (39), we obtain

$$\epsilon_S = \limsup_{n \rightarrow \infty} \frac{\log^+ [\text{Var}_Q I^S(n) / \mathbf{E}_Q I^Q(n)]}{\log n}.$$

In the following, we have

$$\begin{aligned} \text{Var}_Q I^S(n) &\in \left[\left(\sqrt{\text{Var}_Q I^Q(n)} - \sqrt{\text{Var}_Q(I^S(n) - I^Q(n))} \right)^2, \right. \\ &\quad \left. \left(\sqrt{\text{Var}_Q I^Q(n)} + \sqrt{\text{Var}_Q(I^S(n) - I^Q(n))} \right)^2 \right]. \end{aligned}$$

Thus, to show $\epsilon_S = \epsilon_Q$ it suffices to prove that

$$\limsup_{n \rightarrow \infty} \frac{\log^+ \text{Var}_Q(I^S(n) - I^Q(n))}{\log n} = 0.$$

To demonstrate the latter fact, we will use Shannon-Fano coding (38) and a stronger version of Barron's inequality, viz. [24, Theorem 3.1], namely,

$$Q(-\log P(X_1^n) + \log Q(X_1^n) \leq -m) \leq 2^{-m}, \quad (40)$$

which holds for an arbitrary code P . Hence we obtain

$$Q(|I^S(n) - I^Q(n)| \geq 4 \log n + C + m) \leq 2^{-m}$$

for a certain constant C . Subsequently, this yields

$$\begin{aligned} \mathbf{E}_Q (I^S(n) - I^Q(n))^2 &\leq (4 \log n + C)^2 + \sum_{m=0}^{\infty} (4 \log n + C + m + 1)^2 2^{-m} \\ &\leq A(\log n)^2 + B \end{aligned}$$

for certain $A, B > 0$, which proves the claim. \square

Now we may prove Theorem 1.

Proof of Theorem 1: Apply Corollary 1 from the previous section and Theorems 6, 7, and 8 from this section. \square

Although Theorem 1 does not refer to Kolmogorov complexity, an open question remains how parameter ϵ_Q can be evaluated in nontrivial cases (i.e., for a process not being a memoryless source). In Section IV, we will exhibit two processes for which $\delta_Q^+ = \gamma_Q^+$ and $\delta_Q^- = \gamma_Q^-$. Our evaluation of Hilberg exponents for these processes is direct, without bounding the parameter ϵ_Q . We are not aware of any process for which $\delta_Q^+ > \gamma_Q^+$ or $\delta_Q^- > \gamma_Q^-$.

IV Exponents for particular sources

Hilberg exponents can be effectively evaluated in certain cases. In this section we shall compute exponents γ_Q^\pm and δ_Q^\pm related to the underlying measure Q of the process. For IID processes, these Hilberg exponents are trivially equal zero since there is no dependence in the process. Equalities $\delta_Q^\pm = 0$ hold also for Markov processes over a finite alphabet and hidden Markov processes with a finite number of hidden states, since the expected mutual information is bounded for measures of those processes by the data-processing inequality. Hence, in that case, we also have $\gamma_Q^\pm = 0$ by Theorem 1.

Some simple example of a process with unbounded mutual information is the mixture of Bernoulli processes over the alphabet $\mathbb{X} = \{0, 1\}$, which we will call the mixture Bernoulli process:

$$Q(x_1^n) = \int_0^1 \theta^{\sum_{i=1}^n x_i} (1 - \theta)^{n - \sum_{i=1}^n x_i} d\theta = \frac{1}{n+1} \binom{n}{\sum_{i=1}^n x_i}^{-1}. \quad (41)$$

Although X_i are dependent for this measure Q , we will show that the related Hilberg exponents also vanish.

It should be noted that the mixture Bernoulli process is a conditionally IID 1-parameter source. The pointwise mutual information $I^Q(n)$ for conditionally IID sources is equal to a difference of redundancies. Moreover, as shown in [3, 4, 5], for k -parameter processes, the redundancy is proportional to $k \log n$. Formally, this suffices to prove that the Hilberg exponents for the mixture Bernoulli process are zero. Nevertheless, we feel it may be better to present a complete calculation, which is not that long. By the results of [5], our reasoning can be generalized to mixtures of k -parameter exponential families but we skip this topic to present a simple example in a sufficient detail.

For the direct evaluation of the Hilberg exponents, it is convenient to introduce a few further notations. Let the (expected) entropy of a random variable X be written as

$$H_Q(X) = \mathbf{E}_Q [-\log Q(X)], \quad (42)$$

whereas the (expected) mutual information between variables X and Y will be written as

$$I_Q(X; Y) = H_Q(X) + H_Q(Y) - H_Q(X, Y). \quad (43)$$

Moreover, we define the partial sums

$$T_n = \sum_{i=-n+1}^0 X_i, \quad (44)$$

$$S_n = \sum_{i=1}^n X_i. \quad (45)$$

Now we can state the following result for the expected Hilberg exponents.

Theorem 9 *For measure (41), we have $\delta_Q^+ = \delta_Q^- = 0$.*

Proof: It can be easily shown that X_{-n+1}^0 and X_1^n are conditionally independent given T_n and S_n . Hence

$$I^Q(n) = -\log \frac{Q(T_n)Q(S_n)}{Q(T_n, S_n)} \quad (46)$$

so the expected mutual information equals $\mathbf{E}_Q I^Q(n) = I_Q(T_n; S_n)$. Variable S_n assumes under Q each value in $\{0, 1, \dots, n\}$ with equal probability $(n+1)^{-1}$. Hence $0 \leq I_Q(T_n; S_n) \leq H_Q(S_n) = \log(n+1)$, which implies the claim. \square

The random Hilberg exponents γ_Q^\pm for the mixture Bernoulli process also vanish. This follows from $\delta_Q^\pm = 0$ by Theorem 1. It may be insightful, however, to compute γ_Q^\pm directly, following the calculation scheme in [3, 4, 5].

Theorem 10 *For measure (41), $\gamma_Q^+ = \gamma_Q^- = 0$ holds Q -almost surely.*

Proof: Measure Q defined in (41) is not ergodic. Its random ergodic measure (24) takes values of the IID measures

$$E(x_1^n) = \theta^{\sum_{i=1}^n x_i} (1 - \theta)^{n - \sum_{i=1}^n x_i},$$

where θ is a random variable uniformly distributed on $(0, 1)$. Now we will show that $\gamma_Q^+ = \gamma_Q^- = 0$ holds E -almost surely for any θ , which implies that $\gamma_Q^+ = \gamma_Q^- = 0$ holds Q -almost surely. For this aim we will use the Stirling approximation

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n (1 + o(1)).$$

Hence the logarithm of the binomial coefficient is

$$\binom{n}{k} = \frac{1}{2} \log \frac{1}{2\pi} + \frac{1}{2} \log \frac{n}{k(n-k)} + nH\left(\frac{k}{n}\right) + o(1),$$

where $H(p) = -p \log p - (1-p) \log(1-p)$ is the entropy of probability distribution

$(p, 1 - p)$. Thus we obtain

$$\begin{aligned}
I^Q(n) &= \log \frac{(n+1)^2}{2n+1} + \log \frac{\binom{n}{T_n} \binom{n}{S_n}}{\binom{2n}{T_n+S_n}} \\
&= \log \frac{(n+1)^2}{2n+1} + \frac{1}{2} \log \frac{1}{2\pi} \\
&\quad + \frac{1}{2} \log \frac{n}{T_n(n-T_n)} + \frac{1}{2} \log \frac{n}{S_n(n-S_n)} \\
&\quad - \frac{1}{2} \log \frac{n}{(T_n+S_n)(n-T_n-S_n)} \\
&\quad + nH\left(\frac{T_n}{n}\right) + nH\left(\frac{S_n}{n}\right) - 2nH\left(\frac{T_n+S_n}{2n}\right) \\
&\quad + o(1).
\end{aligned}$$

The sequel is straightforward. Define the partial sums $T_n = \sum_{i=-n+1}^0 X_i$ and $S_n = \sum_{i=1}^n X_i$. Quotients S_n/n and T_n/n converge to θ E -almost surely. Further, we may use the Taylor expansion

$$H\left(\frac{T_n}{n}\right) = H(\theta) + H'(\theta) \left(\frac{T_n}{n} - \theta\right) + \frac{1}{2} H''(\theta_1) \left(\frac{T_n}{n} - \theta\right)^2,$$

where $\theta_1 \in [\frac{T_n}{n}, \theta]$, and its analogues for other entropies. This yields

$$\begin{aligned}
I^Q(n) &= \log \frac{(n+1)^2}{2n+1} + \frac{1}{2} \log \frac{1}{2\pi\theta(1-\theta)} \\
&\quad + \frac{1}{2} nH''(\theta_1) \left(\frac{T_n}{n} - \theta\right)^2 + \frac{1}{2} nH''(\theta_2) \left(\frac{S_n}{n} - \theta\right)^2 \\
&\quad - nH''(\theta_3) \left(\frac{T_n+S_n}{2n} - \theta\right)^2 + o(1).
\end{aligned}$$

By the law of the iterated logarithm,

$$\limsup_{n \rightarrow \infty} \frac{|T_n - n\theta|}{\sqrt{n \log \log n}} = C$$

for a certain $C \in (0, \infty)$ holds E -almost surely, and we have similar laws for S_n and $T_n + S_n$. Hence

$$\limsup_{n \rightarrow \infty} \left| I^Q(n) - \log \frac{(n+1)^2}{2n+1} - \frac{1}{2} \log \frac{1}{2\pi\theta(1-\theta)} \right| \leq A \log \log n$$

for a certain $A \in (0, \infty)$. Thus $\gamma_Q^+ = \gamma_Q^- = 0$ holds E -almost surely for any θ . \square

In the next example we will exhibit a process for which Hilberg exponents do not vanish. This process, introduced in [6, 7] under the name of a Santa Fe process is also conditionally IID (nonergodic) but does not constitute a k -parameter source. Its construction is partly motivated linguistically. Namely, we have certain statements X_i that describe for randomly selected indices $K_i = k$

the values of some random binary variables Z_k , where the set of available indices is countably infinite, $k \in \mathbb{N}$.

Formally, the Santa Fe process $(X_i)_{i \in \mathbb{Z}}$ is a sequence of variables X_i which consist of pairs

$$X_i = (K_i, Z_{K_i}), \quad (47)$$

where processes $(K_i)_{i \in \mathbb{Z}}$ and $(Z_k)_{k \in \mathbb{N}}$ are independent and distributed as follows. First, variables Z_k are binary and uniformly distributed,

$$Q(Z_k = 0) = Q(Z_k = 1) = 1/2, \quad (Z_k)_{k \in \mathbb{N}} \sim \text{IID}. \quad (48)$$

Second, variables K_i obey the power law

$$Q(K_i = k) = k^{-1/\beta} / \zeta(\beta^{-1}), \quad (K_i)_{i \in \mathbb{Z}} \sim \text{IID}, \quad (49)$$

where $\beta \in (0, 1)$ is a parameter and $\zeta(x) = \sum_{k=1}^{\infty} k^{-x}$ is the zeta function. Let us note that, formally, random variable $Y = \sum_{k=1}^{\infty} 2^{-k} Z_k$ could be considered a single random real parameter of the process but the distribution of the process $(X_i)_{i \in \mathbb{Z}}$ is not a differentiable function of this parameter. For this reason the Santa Fe process is not a 1-parameter source.

Like in the case of the mixture Bernoulli process, the Hilberg exponents for the Santa Fe process are all equal but, unlike the case of the mixture Bernoulli process, they do not vanish. Their common value is the parameter β in the distribution (49).

Theorem 11 *For process (47), we have $\delta_Q^+ = \delta_Q^- = \beta$.*

Proof: By [7, Proposition 1], $\mathbf{E}_Q I^Q(n)$ grows proportionally to n^β . This implies the claim. \square

Theorem 12 *For process (47), $\gamma_Q^+ = \gamma_Q^- = \beta$ holds Q -almost surely.*

Proof: By Theorems 1 and 11 it suffices to prove that $\gamma_Q^- \geq \beta$. Let $V(k_1^n)$ denote the set of distinct values in sequence k_1^n . We have

$$Q(X_1^n) = Q(K_1^n) 2^{-\text{card } V(K_1^n)}.$$

Hence

$$\begin{aligned} I^Q(n) &= \text{card } V(K_{-n+1}^0) + \text{card } V(K_1^n) - \text{card } V(K_{-n+1}^n) \\ &= \text{card}(V(K_{-n+1}^0) \cap V(K_1^n)). \end{aligned}$$

Let $L_n = n^{\beta(1-\epsilon)}$, where $\epsilon > 0$. We have

$$\begin{aligned}
& Q(\{1, 2, \dots, \lfloor L_n \rfloor\} \not\subset V(K_{-n+1}^0) \cap V(K_1^n)) \\
& \leq \sum_{k=1}^{\lfloor L_n \rfloor} Q(k \notin V(K_{-n+1}^0) \cap V(K_1^n)) \\
& \leq \sum_{k=1}^{\lfloor L_n \rfloor} [Q(k \notin V(K_{-n+1}^0)) + Q(k \notin V(K_1^n))] \\
& = \sum_{k=1}^{\lfloor L_n \rfloor} 2(1 - Q(K_i = k))^n \\
& \leq 2L_n \left[1 - \frac{L_n^{-1/\beta}}{\zeta(\beta^{-1})} \right]^n \\
& = 2L_n \exp \left[n \ln(1 - L_n^{-1/\beta} / \zeta(\beta^{-1})) \right] \\
& \leq 2L_n \exp \left[-n L_n^{-1/\beta} / \zeta(\beta^{-1}) \right] \\
& \leq 2n^\beta \exp \left[-n^\epsilon / \zeta(\beta^{-1}) \right].
\end{aligned}$$

Since

$$\sum_{n=1}^{\infty} Q(\{1, 2, \dots, \lfloor L_n \rfloor\} \not\subset V(K_{-n+1}^0) \cap V(K_1^n)) < \infty,$$

hence, by the Borel-Cantelli lemma, sets $\{1, 2, \dots, \lfloor L_n \rfloor\}$ are Q -almost surely subsets of $V(K_{-n+1}^0) \cap V(K_1^n)$ for all but finitely many n . In consequence, $I^Q(n) \geq \lfloor n^{\beta(1-\epsilon)} \rfloor$ for those n , which implies $\gamma_Q^- \geq \beta$ since ϵ was chosen arbitrarily. \square

It should be noted that both the measures of the mixture Bernoulli process and the Santa Fe processes are nonergodic and computable. Hence Corollaries 2 and 3 apply to these sources. There exist also mixing (i.e., ergodic in particular) and computable measures Q for which exponents $\delta_Q^+ = \delta_Q^-$ assume an arbitrary value in $(0, 1)$. These processes can be constructed as a modification of the original Santa Fe process (47). For the construction, see [7].

The third example will be a process for which the upper and the lower expected Hilberg exponents are different. The process is a slight modification of the Santa Fe process, though different than that discussed in [7]. Consider a sequence of fixed numbers $(a_k)_{k \in \mathbb{N}}$ where $a_k \in \{0, 1\}$. Let

$$X_i = (K_i, Y_{K_i}), \quad (50)$$

where $Y_k = a_k Z_k$, whereas processes $(K_i)_{i \in \mathbb{Z}}$ and $(Z_k)_{k \in \mathbb{N}}$ are independent and distributed as for the original Santa Fe process. If $a_k \neq 0$ for some k , process (50) is also nonergodic.

Theorem 13 *There exists such a sequence $(a_k)_{k \in \mathbb{N}}$ that for process (50), we have $\delta_Q^+ = \beta$ and $\delta_Q^- = 0$.*

Proof: Analogously as in the proof of [7, Proposition 1], we obtain

$$\mathbf{E}_Q I^Q(n) = \sum_{k=1}^{\infty} a_k \left(1 - \left(1 - \frac{A}{k^{1/\beta}} \right)^n \right)^2,$$

where $A := 1/\zeta(\beta^{-1})$. In case of the original Santa Fe process, we have $a_k = 1$ for all k and then $\mathbf{E}_Q I^Q(n)$ is asymptotically proportional to n^β [7, Proposition 1]. Thus, it is sufficient to show that $\delta_Q^+ \geq \beta$ and $\delta_Q^- \leq 0$ for a certain sequence $(a_k)_{k \in \mathbb{N}}$.

We have two bounds

$$\begin{aligned} \left(1 - \frac{A}{k^{1/\beta}} \right)^n &\geq \max \left\{ 0, \frac{nA}{k^{1/\beta}} \right\}, \\ \left(1 - \frac{A}{k^{1/\beta}} \right)^n &= \exp \left(n \ln \left(1 - \frac{A}{k^{1/\beta}} \right) \right) \leq \exp \left(-\frac{nA}{k^{1/\beta}} \right). \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{E}_Q I^Q(n) &\leq \sum_{k=1}^{\lfloor n^{2\beta} \rfloor} a_k + \sum_{k=\lfloor n^{2\beta} \rfloor + 1}^{\infty} a_k \left(\frac{nA}{k^{1/\beta}} \right)^2 \\ &\leq \sum_{k=1}^{\lfloor n^{2\beta} \rfloor} a_k + 1 + A^2 n^2 \int_{n^{2\beta}}^{\infty} \frac{1}{k^{2/\beta}} dk \\ &= \sum_{k=1}^{\lfloor n^{2\beta} \rfloor} a_k + 1 + A^2 n^2 \frac{\beta}{2-\beta} (n^{2\beta})^{\frac{\beta-2}{\beta}} \leq \sum_{k=1}^{\lfloor n^{2\beta} \rfloor} a_k + \left(1 + \frac{A^2 \beta}{2-\beta} \right), \\ \mathbf{E}_Q I^Q(n) &\geq \sum_{k=1}^{\infty} a_k \left(1 - \exp \left(-\frac{nA}{k^{1/\beta}} \right) \right) \geq \sum_{k=1}^{\lfloor n^\beta \rfloor} a_k (1 - \exp(-A)). \end{aligned}$$

Having these two auxiliary results, we can easily show that $\delta_Q^+ \geq \beta$ and $\delta_Q^- \leq 0$ for the following sequence $(a_k)_{k \in \mathbb{N}}$. Let $(b_m)_{m \in \mathbb{N}}$ and $(c_m)_{m \in \mathbb{N}}$ be two sequences of natural numbers, where additionally $c_0 = 0$ and

$$\lfloor c_{m-1}^\beta \rfloor < \lfloor b_m^{2\beta} \rfloor < \lfloor c_m^\beta \rfloor$$

for all m , and let $(\epsilon_m)_{m \in \mathbb{N}}$ be a sequence of real numbers $\epsilon_m = \beta/m$. We put

$$a_k = \begin{cases} 0, & \lfloor c_{m-1}^\beta \rfloor < k \leq \lfloor b_m^{2\beta} \rfloor, \\ 1, & \lfloor b_m^{2\beta} \rfloor < k \leq \lfloor c_m^\beta \rfloor. \end{cases}$$

As for sequences $(b_m)_{m \in \mathbb{N}}$ and $(c_m)_{m \in \mathbb{N}}$, we choose them to satisfy

$$\begin{aligned} \lfloor c_{m-1}^\beta \rfloor + \left(1 + \frac{A^2 \beta}{2-\beta} \right) &\leq b_m^{\epsilon_m}, \\ (\lfloor c_m^\beta \rfloor - \lfloor b_m^{2\beta} \rfloor) (1 - \exp(-A)) &\geq c_m^{\beta - \epsilon_m}. \end{aligned}$$

In this way we obtain

$$\begin{aligned}\mathbf{E}_Q I^Q(b_m) &\leq \lfloor c_{m-1}^\beta \rfloor + \left(1 + \frac{A^2\beta}{2-\beta}\right) \leq b_m^{\epsilon_m}, \\ \mathbf{E}_Q I^Q(c_m) &\geq (\lfloor c_m^\beta \rfloor - \lfloor b_m^{2\beta} \rfloor) (1 - \exp(-A)) \geq c_m^{\beta-\epsilon_m}.\end{aligned}$$

Hence $\delta_Q^+ \geq \beta$ and $\delta_Q^- \leq 0$, as requested. \square

Evaluation of random Hilberg exponents for process (50) seems difficult.

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